Introduction to the Lattice Boltzmann Method

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Taylor-Green vortex decay

In what follows we will examine how to simulate the Taylor-Green vortex decay, which is prescribed the following set of equations:

$$
\begin{aligned}
u_x(x, y) = &\quad u_0 \sin(x) \sin(y) \exp(-2\nu t) \\
u_y(x, y) = &\quad u_0 \cos(x) \cos(y) \exp(-2\nu t) \\
p(x, y) = &\quad p_0 + 0.25 u_0^2 \big(\cos(2x) - \cos(2y)\big) \exp(-4\nu t) \\
&\quad 0 < x, y < 2\pi
\end{aligned}
$$

One can check that the expressions for velocities and pressure do satisfy the incompressible Navier-Stokes equation:

$$
\begin{aligned}
\partial_\alpha u_\alpha &= 0 \\
\rho \partial_t u_\alpha \rho \partial_\beta u_\alpha u_\beta &= -\frac{1}{3} \partial_\alpha \rho + \nu \partial^2_{\beta\beta} u_\alpha
\end{aligned}
$$

EXERCISE 1

To implement the lattice Boltzmann algorithm for the Taylor-Green vortex and visualize it with the Matlab:

1. To implement the initialization with the equilibrium function
2. To construct the equilibrium function
3. To implement the collision operator
4. To implement the streaming operator
5. To visualize the $u_x\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ with plot Matlab function and compare it with the analytical solution.

Note that to compare it with the analytical solution one needs to understand that the real viscosity of the problem is $\Delta t \frac{1}{3}(\tau - \frac{1}{2})$ and time should be transfered to the real system as $\Delta t = \frac{u_{LB}}{u_{phys}} \Delta x$

Convergence

Exercise 2

Examine that then nice parameters are taken then the method is second order convergent. For this purpose one needs to take consequent lattice grids $NX = NY = 16$,$NX = NY = 32$, $NX = NY = 64$,$NX = NY = 128$ and examine the error with the analytical solution in L_2 norm sense which is defined as $E = \sqrt{\frac{\int |u(x,y) - u_{theor}(x,y)|^2 \mathrm{d}V}{V}} = \sqrt{\frac{\sum_i |u(x_i,y_i) - u_{theor}(x_i,y_i)|^2}{NX\,NY}}$. As far as grid cannot be changed separately from the time one needs to perform scaling analysis to take the amplitude in different time steps (as well notice that the viscosity as well proportional to Δt):

$$\Delta t \propto \Delta x \propto \frac{1}{NX};\ \nu \propto \Delta t$$

Thus if the error is examined for $NX = 16$ at $t = 100$, then for $NX = 32$ there should be $t = 400$ iterations performed.

The exercise consists in the following:

1. Write down L_2 error for different grids $NX = NY = 16$,$NX = NY = 32$, $NX = NY = 64$,$NX = NY = 128$ and corresponding times $N_{steps} = 100$, $N_{steps} = 400$, etc.
2. Check whether the errors fall on second order convergent curve in log-log scale.

One can do study with different parameters (critical one):

EXERCISE 3

In what follows we will examine a number of different parameters to see the limitations of the lattice Boltzmann method:

- We will start with the Mach number influence. Fix the following set of parameters $NX = NY = 32$, $NSTEPS = 400$ and increase velocity starting from 0.001 to 0.005, 0.01,0.05,0.1,0.2,0.3,0.4,0.5. See results on a plot. Especially look at the beginning of the plot with a zoom tool.
- The influence of decreasing τ. The same dimensions and $u_0 = 0.001$ but with $\tau = 1.0$,$\tau = 0.8$,$\tau = 0.6$,$\tau = 0.55$, $\tau = 0.51$, $\tau = 0.505$, $\tau = 0.49$
- The influence of increasing τ. The dimensions $NX = NY = 64$ and $NSTEPS = 400$. Try $\tau = 1.0$, $\tau = 2.0$,$\tau = 5.0$, $\tau = 10.0$. Examine the difference in the beginning.

Four rows mill

A modification of the Taylor-Green flow is so-called four rows mill. It is the steady state solution of the following system:

$$\partial_\alpha u_\alpha = 0$$

$$\rho\partial_t u_\alpha \rho\partial_\beta u_\alpha u_\beta = -\frac{1}{3}\partial_\alpha p + F_\alpha + \nu\partial^2_{\beta\beta} u_\alpha$$

if the force is in the form:

$$F_x = 2\nu \sin(x)\sin(y)$$

$$F_y = 2\nu \cos(x)\cos(y),$$

then the solution will have the same form as in the Taylo-Green case but without the time dependancy:

$$u_x(x,y) = \quad u_0 \sin(x)\sin(y)$$

$$u_y(x,y) = \quad u_0 \cos(x)\cos(y)$$

$$p(x,y) = \quad p_0 + 0.25u_0^2\big(\cos(2x) - \cos(2y)\big)$$

This is the excellent benchmark to check the force implementation.

EXERCISE 1

Perform the following steps:

- Implement the Guo forcing and add it to the Green-Taylor vortex implementation. To remind how the force looks like:

$$F_i = w_i \left(1 - \frac{\omega}{2}\right)(3(\boldsymbol{c_i} - \boldsymbol{u}^m) + 9\boldsymbol{c_i}(\boldsymbol{c_i} \cdot \boldsymbol{u}^m)) \cdot \boldsymbol{F},$$

  where the macroscopic velocity can be calculated from the expression $\rho \boldsymbol{u}^m = \sum_i f_i \boldsymbol{c_i} + \boldsymbol{F}/2$.
- Examine the influence of small ω or large τ. Take the following values: $\tau = 1$, $\tau = 2$, $\tau = 5$, $\tau = 10$. See how the max amplitude behaves and corresponding error.
- Perform two tests with $\tau = 10$ ($NX = NY = 64$ and $NX = NY = 128$). See how the resolution influences. Basically the Knudsen number from physical point of view is proportional to the Mach number multiplied by the inverse Reynolds number $\frac{Ma}{Re}$. Thus, inreasing the viscosity can be cured by the increasing of the resolution.

TRT MODEL

Implement the TRT model:

1. Do the projection of the distribution functions (usual and equilibrium) on symmetrical parts:
$$f_i^+ = \frac{f_i + f_{\bar{i}}}{2},\ f_i^- = \frac{f_i - f_{\bar{i}}}{2}$$
2. Do the implementation of the force. Hint the same as Guo but ω is substituted with ω_-
3. Perform the collision: $f_i^* = f_i - \omega_-(f_i^- - f_i^{eq,-}) - \omega_+(f_i^+ - f_i^{eq,+})$
4. Check the result for $\omega = 0.1$. See the difference with the BGK :)