Corner Boundary Conditions
in lattice Boltzmann method

Goncalo Silva
Department of Mechanical Engineering
Instituto Superior Técnico (IST)
Lisbon, Portugal

Outline

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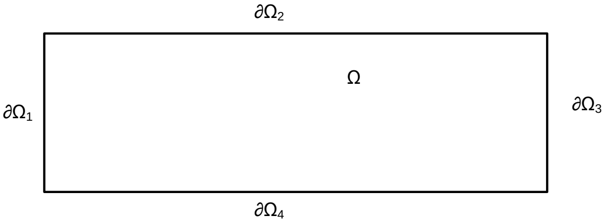
Definitions

- Boundary value problem geometry:



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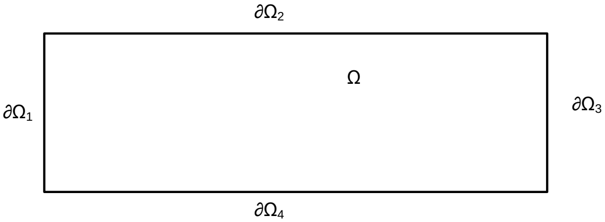

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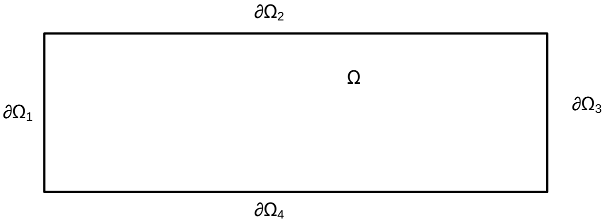

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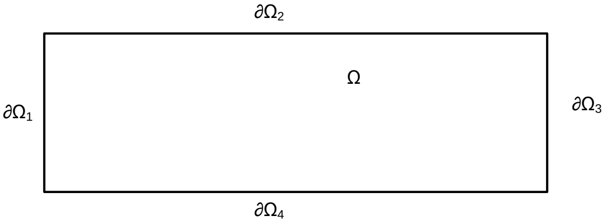

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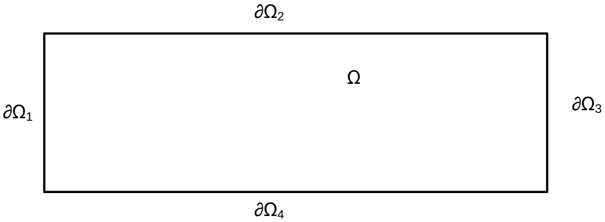

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- Particularly important when *e.g.* $\varphi(\partial\Omega_1) \neq \varphi(\partial\Omega_2)$
- Smooth evolution of φ on $\partial\Omega$
- Avoid singularities of φ on $\partial\Omega$

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- Isothermal and incompressible Navier-Stokes equations

$$\left\{ \begin{array}{r} (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\triangle\mathbf{u} + \mathbf{a} \\ \nabla\cdot\mathbf{u} = 0 \end{array} \right. \quad \text{in } \Omega$$

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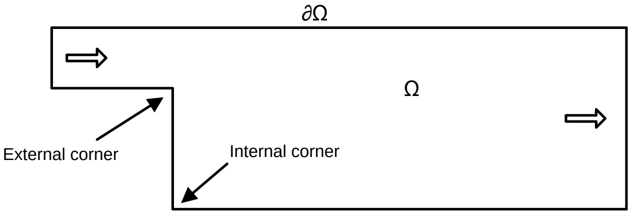
- Everything else follows the same philosophy of boundary conditions

Examples

- Examples of flow domains with corners:

Backward facing step flow



Examples

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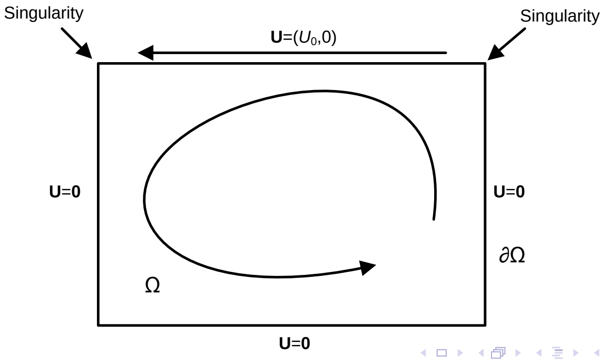
Lid-driven cavity flow



Outline

1. Introduction
   - Corners in Boundary Value Problems
2. Motivation
   - Navier-Stokes Boundary Conditions: Corners
3. Lattice Boltzmann Boundary Conditions: Corners
   - Problem definition
   - Particulate dynamics
   - Using Chapman-Enskog

Introduction and motivation

- Lattice Boltzmann method (LBM)

$$f_\alpha(\mathbf{x} + \mathbf{c}_\alpha \Delta t, t + \Delta t) = f_\alpha(\mathbf{x}, t) - \omega(f_\alpha - f_\alpha^{(eq)})|_{(\mathbf{x},t)} \quad \text{in } \Omega$$

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  → f_α set in a higher DoF system than $\{\rho, \mathbf{u}, \mathbf{\Pi}\}$, hence:

    - Trivial: $f_\alpha \longrightarrow \{\rho, \mathbf{u}, \mathbf{\Pi}\}$
    - Complex: $\{\rho, \mathbf{u}, \mathbf{\Pi}\} \longrightarrow f_\alpha$

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  → Incorrect upscaling ⟶ Unwanted behavior, e.g. Knudsen layers

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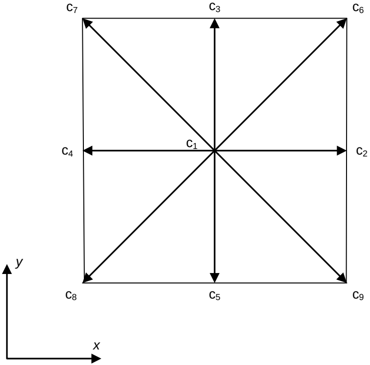
  $\downarrow$ Inaccurate corner implementation may contaminate the solution everywhere in the domain

  $\downarrow$ Interpolation or extrapolation may not be possible at corners

Lattice structure

...remember!

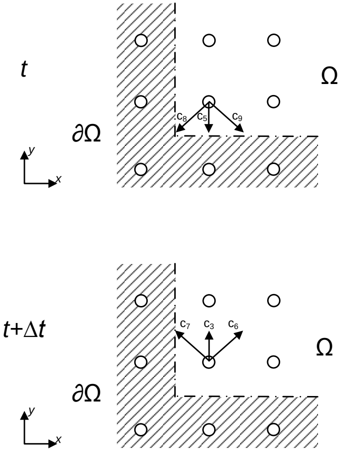


- D2Q9 model

$$\begin{aligned}
\mathbf{c}_1 &= (0,0)\\
\mathbf{c}_2 &= (1,0)\\
\mathbf{c}_3 &= (0,1)\\
\mathbf{c}_4 &= (-1,0)\\
\mathbf{c}_5 &= (0,-1)\\
\mathbf{c}_6 &= (1,1)\\
\mathbf{c}_7 &= (-1,1)\\
\mathbf{c}_8 &= (-1,-1)\\
\mathbf{c}_9 &= (-1,-1)
\end{aligned}$$

Bounceback on **corner**

Bottom-left Corner



Bounceback on **corner**

- **Pros**
  - → Mass is exactly conserved
  - → Stable for ω close to 2 (i.e. for high Re)
  - → Local
  - → Flexibility in handling wall, edges, corners both in 2D and 3D
  - → Very simple to implement from a programming viewpoint
- **Cons**
  - → Velocity accuracy may decrease from 2nd to 1st
  - → Pressure accuracy may decrease from 1st to 0th
  - → In SRT model momentum is not exactly conserved (viscosity dependent slip velocity)

Bounceback on **corner**

Question:

Apply the half-way bounceback scheme to **top-left** and **bottom-right** corners

Zou He on **corner**

- $\rightarrow$ Boundary node and solid node coincide
- $\rightarrow$ Only unknown incoming populations are modified
- $\rightarrow$ Set ρ or $\mathbf{u}$ in $f_\alpha^{(0)}(\rho, \mathbf{u})$
- $\rightarrow$ Construct $f_\alpha^{(1)}$ from the symmetry requirement

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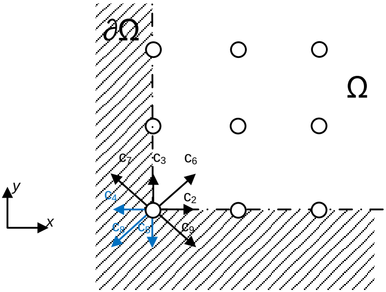
$\rightarrow$ Additional problem: the so-called "buried links"

Zou He on **Corners**



- Known:
  - $\rightarrow$ $\mathbf{u} = \mathbf{0}$
  - $\rightarrow$ $f_\alpha = (f_4, f_5, f_8)$

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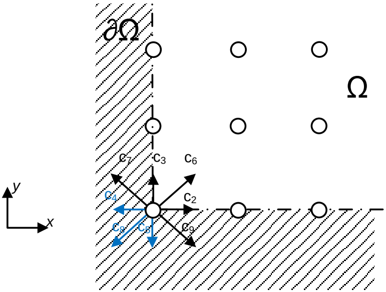

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- Unknown (6 variables):
  - $\rightarrow$ ρ
  - $\rightarrow$ $f_\alpha = (f_2, f_3, f_6, f_7, f_9)$

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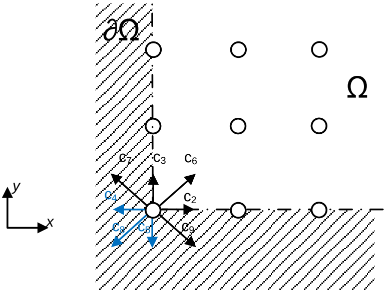

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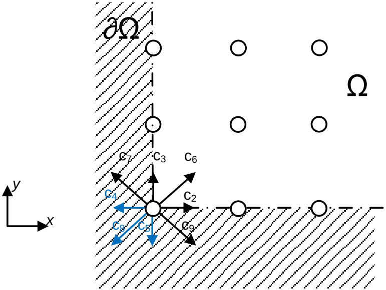
- Unknown (6 variables):
  - $\rightarrow$ ρ
  - $\rightarrow$ $f_\alpha = (f_2, f_3, f_6, f_7, f_9)$
- 3 Equations (2 linearly independent):
  - $\rightarrow$ $\sum f_\alpha = \rho$
  - $\rightarrow$ $\sum \mathbf{c}_\alpha f_\alpha = \mathbf{u}$

Zou He on **corner**

1) Computing ρ...



- Population velocity set at corner node:

  $\rightarrow C_+ = \{\mathbf{c}_3, \mathbf{c}_6, \mathbf{c}_7\}$

  $\rightarrow C_0 = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$

  $\rightarrow C_- = \{\mathbf{c}_5, \mathbf{c}_8, \mathbf{c}_9\}$

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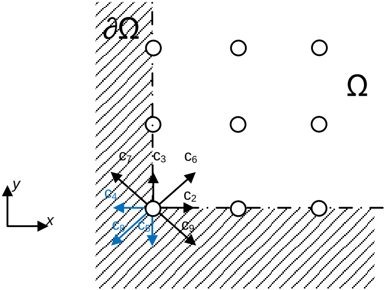

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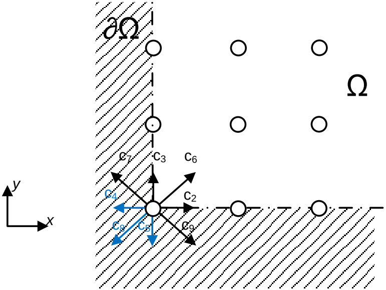

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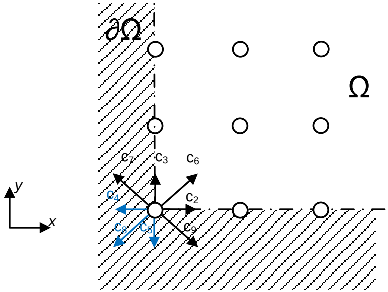
- **Not possible** to compute ρ as in boundary planes
- Inevitable **solution**: extrapolate ρ from adjacent nodes

Zou He on **corner**

2) Computing $\{f_2, f_3, f_6, f_7, f_9\}$**...**



- Symmetry of $f_\alpha^{(1)}$(3 equations):
  - → Bounceback of non-equilibrium populations

Zou He on **corner**

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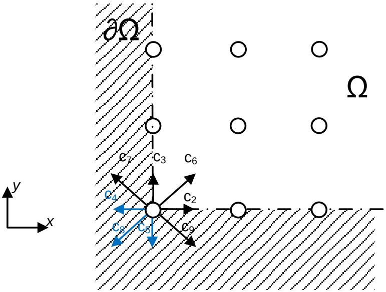

- Symmetry of $f_\alpha^{(1)}$(3 equations):
  - $\rightarrow$ Bounceback of non-equilibrium populations
- Still 2 unknowns left:
  - $\rightarrow$ Populations of buried link $\{f_7, f_9\}$

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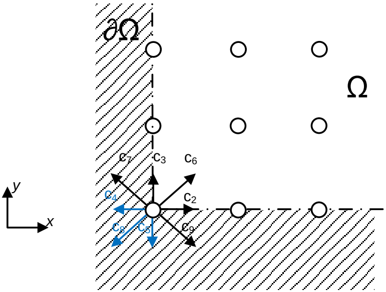

- Symmetry of $f_\alpha^{(1)}$(3 equations):
  - → Bounceback of non-equilibrium populations
- Still 2 unknowns left:
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- Problem underspecified:
  - → 5 eqs. and 6 unknowns

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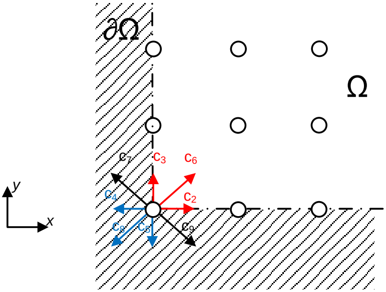

- Non-equilibrium bounceback:

$$f_2 - f_2^{(0)} = f_4 - f_4^{(0)}$$

$$f_3 - f_3^{(0)} = f_5 - f_5^{(0)}$$

$$f_6 - f_6^{(0)} = f_8 - f_8^{(0)}$$

Zou He on **corner**

2) Computing $\{f_2, f_3, f_6, f_7, f_9\}$**....**

- Solution for some of the unknown incoming populations (from non-equilibrium bounceback):

$$f_2 = f_4 + \frac{2}{3}u_x$$

$$f_3 = f_5 + \frac{2}{3}u_y$$

$$f_6 = f_8 + \frac{1}{6}(u_x + u_y)$$

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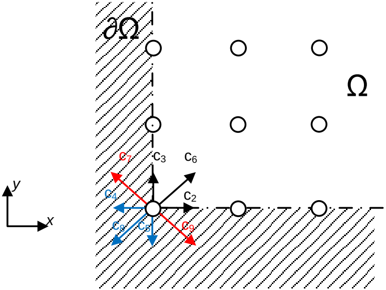

- Buried populations:

$$u_x = (f_2 + f_6 + f_9) - (f_4 + f_7 + f_8)$$

$$u_y = (f_3 + f_6 + f_7) - (f_5 + f_8 + f_9)$$

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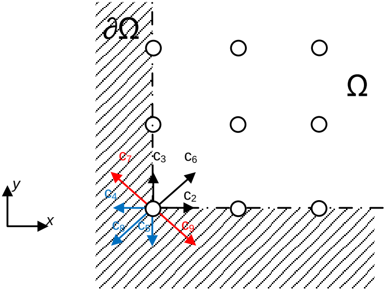

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$$u_y = (f_3 + f_6 + f_7) - (f_5 + f_8 + f_9)$$

- Possible solution:

$$f_7 = \frac{1}{12}(u_y - u_x)$$

$$f_9 = \frac{1}{12}(u_x - u_y)$$

Zou He on **corner**

2) Computing f_1....

- Recall density has been extrapolated. Therefore, to ensure proper upscaling between macroscopic parameters and LB populations:

  1) Set $f_1 = 0$

  2) Compute $f_1 = \rho - \sum f_\alpha$

Zou He on **corner**

Question:

Apply Zou He boundary condition scheme to **top-left** and **bottom-right** corners

Exercise III

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Backward facing step flow